Adventures in the Quantum Polynomial Ring: Patterns in p-Polynomials

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Definition of p-Polynomials

The p-Polynomials are a method of describing paths within a partial ordering system of symmetric group elements

1. $p_{u,v,w}(q) = 0$ if $w \ngeq u^{-1}v$
2. $p_{u,v,w}(q) = 1$ if $w = u^{-1}v$
3. For every s such that $us < u$,

$$p_{u,v,w}(q) = \begin{cases} p_{us,v,sw}(q) & \text{if } sw > w \\ p_{us,v,sw}(q) + qp_{us,v,w}(q) & \text{otherwise} \end{cases}$$

Where u, v, w are elements of the n^{th} symmetric group.

Background

Symmetric Group

- Every possible permutation of the numbers $1, 2, 3, \cdots, n$, denoted S_n.
- There are a total of $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ elements in S_n.

Simple Reflections

- Changes the order of two consecutive elements.
- Denoted s_i

$$123 \rightarrow 213 \quad \text{after applying } s_1$$
$$123 \rightarrow 231 \quad \text{after applying } s_2 s_1$$

Inverse

- Two identical reflections cancel each other out

$$s_1 s_2 * s_2 = s_1$$

Braid Relation

- Representations for a permutation are not unique due to the braid relation:

$$\text{If } |i - j| = 1 \text{ then } s_i s_j s_i = s_j s_i s_j$$

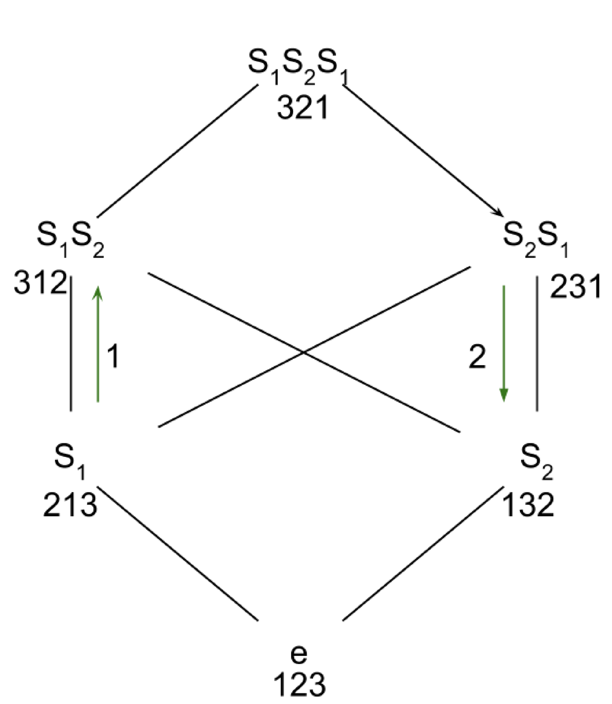
Ascent

- Multiplying a word with a simple reflection such that the resulting word is longer than the original.

Descent

- Multiplying a word with a simple reflection such that the resulting word is shorter than the original.

Bruhat Ordering of the Symmetric Group S_3



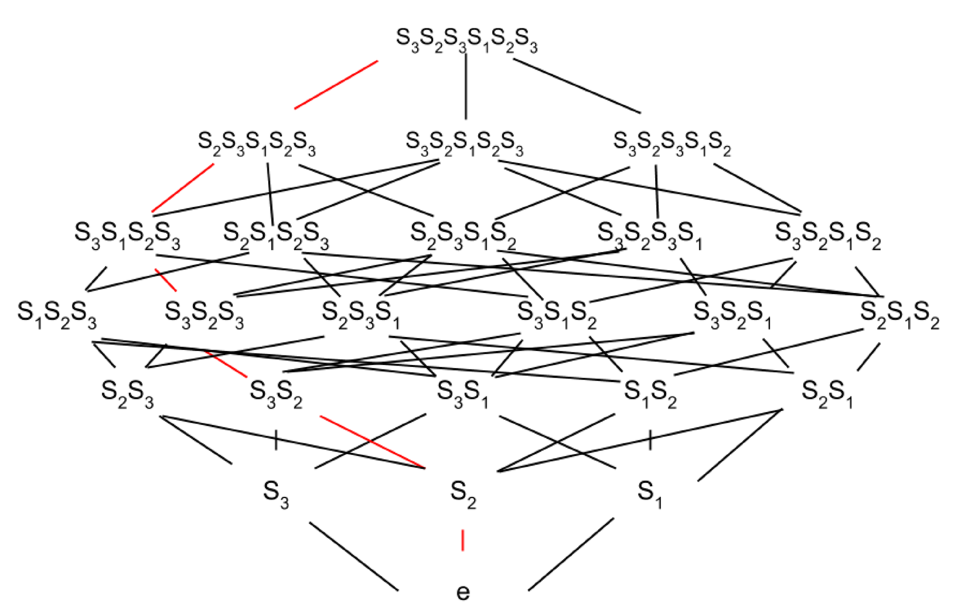
1. An *ascent* can also be understood as moving up in the graph.
2. Similarly, a *descent* can be understood as moving down in the graph.

The Counting Method

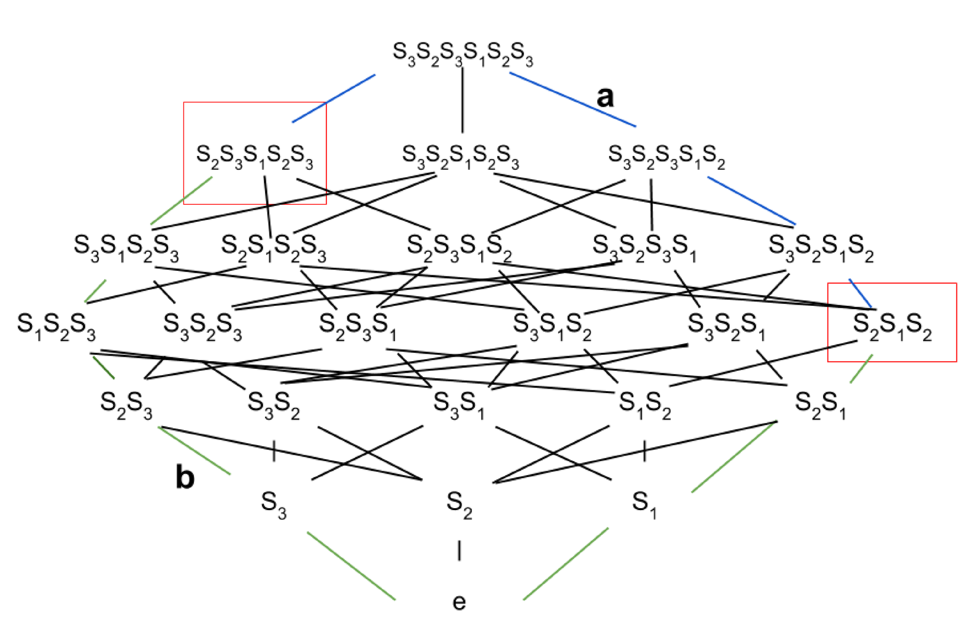
- Begin with any reduced expression for u and v.
- u will be our path between the beginning point, v, and endpoint w.
- Multiply v on the left by the leftmost reflection s in the decomposition of u.
- If $sv > v$ then we draw a single path from v to sv.
- If $sv < v$ then we choose one of two paths, either stay put at v or go down from v to sv.
- Use the rightmost simple reflection of su (or in other words the second simple reflection from the left in the decomposition of u) and repeat the above process.
- The polynomial acquires a variable, q, each time we choose to stay put.
- The resulting polynomial represents every possible method of moving from v to w using path u.

p-Polynomials equal to 1

- Moving on every possible occasion results in a polynomial equal to 1.
- It can be shown that the number of polynomials equal to 1 in each symmetric group is $(n!)^2$.



- k represents the length of the shortest path between the highest and lowest ponts on the graph.
- Every two points can be connected as long as the length of the connecting path is less than k.



- $a + b = 2k$
- Either $a \leq k$ or $b \leq k$

Since a path can be made connecting each permutation to every other permutation and there are $n!$ permutations, the number of polynomials equal to one is $(n!)^2$.

Longest Word p-Polynomial

- The longest word for any symmetric group S_n is represented as $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_{n-1} s_{n-2} \ldots s_2 s_1$
- The single most interesting p-polynomial is the longest word p-polynomial $p_{w_0,w_0,w_0}(q)$
- This polynomial counts the paths that start and end at the top of the Bruhat order graph, following w_0.

Examples of $p_{w_0,w_0,w_0}(q)$

S_2: q
S_3: $q^3 + q$
S_4: $q^6 + 3q^4 + q^2$
S_5 : $q^{10} + 6q^8 + 10q^6 + 5q^4 + q^2$
S_6 : $q^{15} + 10q^{13} + 36q^{11} + 57q^9 + 39q^7 + 9q^5 + q^3$
S_7 : $q^{21} + 15q^{19} + 91q^{17} + 287q^{15} + 505q^{13} + 498q^{11} + 267q^9 + 75q^7 + 12q^5 + q^3$

Patterns

- The highest powers are triangular numbers: $1, 3, 6, 10, 15, \cdots$. This is due to the length of w_0 always being a triangular number.
- The coefficient of the highest power is always 1.
  - The highest power's coefficient represents staying put at w_0 through each recursion, and never leaving. There is 1 way to do this.
- The coefficients of the second highest power are triangular numbers.
  - We then count the consecutive pairs of each simple reflection $s_1, s_2, \cdots s_{n-1}$.
  - This count of pairs is a triangular number.

Example - Symmetric Group S_6

$S_6 : w_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_1$
Consecutive pairs: $s_1 : 4$ $s_2 : 3$ $s_3 : 2$ $s_4 : 1$ $s_5 : 0$
Sum of all pairs: 10
Coefficient of Second highest power: 10

- Let $\phi(n)$ denote the number of nonzero terms in the polynomial $p_{w_0,w_0,w_0}(q)$ in S_n
  - For every higher symmetric group the number of nonzero terms of $p_{w_0,w_0,w_0}(q)$ increases such that $\phi(n) = \phi(n-1) + k$ where k is initially zero
  - For every odd n of S_n, k is incremented by a value of 1
- The coefficient of the lowest power is always 1.
- The only real root of the polynomials $p_{w_0,w_0,w_0}(q)$ is 0. All other roots are complex numbers.

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