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On the Creation of Rank Two Centrosymmetric Matrices

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Abstract

For any square matrix B, we can create a centrosymmetric matrix A = B+JBJ where J is the skew identity matrix. If the matrix B is created as the outer product of two vectors v and h, the resulting centrosymmetric matrix has a maximal rank of 2. However, not all such rank two matrices can be written in this form. In this work, we fully examine when a 3 x 3 centrosymmetric matrix can be created from two vectors and generalize our results to larger matrices.

Matrix as an Outer Product

Let $v = [v_1, v_2, v_3, \ldots v_n]^T$ and $h = [h_1, h_2, h_3, \ldots h_n]$. The outer product $P = v \otimes h$ creates an n $\times$ n rank 1 matrix. If J is the skew identity matrix, JPJ rotates the matrix 180 degrees. Thus, A = P+JPJ is a centrosymmetric matrix with rank ≤ 2 and with $A_{ij} = v_i h_j + v_{n+1-i} h_{n+1-j}$.

3 $\times$ 3 Examples

In the 3 $\times$ 3 case, we have

$$A = \begin{bmatrix} v_1h_1 + v_3h_3 & v_1h_2 + v_3h_2 & v_1h_3 + v_3h_1 \\ v_2h_1 + v_2h_3 & 2v_2h_2 & v_2h_3 + v_2h_1 \\ v_3h_1 + v_1h_3 & v_3h_2 + v_1h_2 & v_3h_3 + v_1h_1 \end{bmatrix}$$

Consider:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Since $A_{22} = 0$ then this implies that $2v_2h_2 = 0$. If $v_2 = 0$ then A_{21} and A_{23} also equal zero, thus the middle row is all zeros. If $h_2 = 0$ then A_{12} and A_{32} also equal zero, thus the middle column is all zeros. Thus, this matrix can't be written as an outer product.

On the other hand, if the matrix contains no zeros, let $v = [A_{13}, A_{21}, A_{11}]^T$ and $h = [0, \frac{A_{12}}{A_{11}+A_{13}}, 1]$. Then:

$$A = \begin{bmatrix} A_{11} & A_{13}\frac{A_{12}}{A_{11}+A_{13}} + A_{11}\frac{A_{12}}{A_{11}+A_{13}} & A_{13} \\ A_{21} & 2A_{21}\frac{A_{12}}{A_{11}+A_{13}} & A_{21} \\ A_{13} & A_{11}\frac{A_{12}}{A_{11}+A_{13}} + A_{13}\frac{A_{12}}{A_{11}+A_{13}} & A_{11} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{21} \\ A_{13} & A_{12} & A_{11} \end{bmatrix}$$

since $A_{13}\left(\frac{A_{12}}{A_{11}+A_{13}}\right) + A_{11}\left(\frac{A_{12}}{A_{11}+A_{13}}\right) = A_{12}\left(\frac{A_{11}+A_{13}}{A_{11}+A_{13}}\right) = A_{12}$

and $2\left(\frac{A_{21}A_{12}}{A_{11}+A_{13}}\right) = \frac{2(v_2h_1 + v_2h_3)(v_1h_2 + v_3h_2)}{v_1h_1 + v_3h_3 + v_1h_3 + v_3h_1} = \frac{2v_2h_2(v_1+v_3)(h_1+h_3)}{(v_1+v_3)(h_1+h_3)} = 2v_2h_2 = A_{22}$

Matrices that can not be created		Matrices that need certain conditions
$A_{22} = 0$,	A_{11} & $A_{12} = 0$	$A_{11} = 0$ if $2A_{12}A_{21} = A_{22}A_{13}$
A_{11} & $A_{13} = 0$,	A_{11} & $A_{21} = 0$	$A_{12} = 0$ if $A_{11} = -A_{13}$
A_{12} & $A_{13} = 0$,	A_{13} & $A_{21} = 0$	$A_{21} = 0$ if $A_{11} = -A_{13}$
A_{11} & $A_{22} = 0$,	A_{13} & $A_{22} = 0$	A_{12} & $A_{21} = 0$ if $A_{11} = -A_{13}$
A_{11} & A_{12} & $A_{21} = 0$		No Zeros if $A_{22} = \frac{2A_{21}A_{12}}{A_{11}+A_{13}}$
A_{12} & A_{13} & $A_{21} = 0$		
A_{11} & A_{13} & $A_{22} = 0$		

Due to the rank two nature of these matrices, the only cases guaranteed to work are those in which a row is either repeated or all zeros.

Generalized Results

We now focus on extending the results of the 3 $\times$ 3 case to rank two centrosymmetric matrices of any general odd size.

For each of the following results, let A be a rank two centrosymmetric matrix of size 2n+1 $\times$ 2n+1 such that A can be written as P+JPJ.

Result 1: If $A_{n+1,n+1} = 0$, then either the n+1st row or n+1st column must only consist of zeros.

Result 2: If there is a zero in the middle column or row but not the middle entry, that row or column displays negative symmetry.

Result 3: Relationships involving the middle row or column

$$(A_{ij} + A_{i,2n+2-j})A_{n+1,n+1} = (A_{n+1,j} + A_{n+1,2n+2-j})A_{i,n+1}$$

Result 4: Relationships not involving the middle row or column

$$(A_{ij} \pm A_{i,2n+2-j})(A_{kl} \pm A_{k,2n+2-l}) = (A_{il} \pm A_{i,2n+2-l})(A_{kj} \pm A_{k,2n+2-j})$$

Future Work

The most obvious place to extend this work is to repeat this process for matrices of size 2n $\times$ 2n.

More interestingly, it is already known that each eigenvalue of a rank two centrosymmetric matrix directly relates to the trace and skew-trace of the matrix. For the case where such a matrix can be deconstructed as the sum of two outer products, this means the eigenvalues directly relate to the two vectors v and h. We would like to explore this relationship in more depth, eventually seeing if we can find results relating the vectors v and h to perturbations of rank two centrosymmetric matrices.

Thanks

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